

Construction of Log-linear and Logit Models via Mutual Information Identity

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Abstract

Association of discrete variables is analyzed using the mutual information of multivariate multinomial distributions. A geometric analysis of conditional mutual information is used to select indispensable predictors and interaction effects for constructing generalized linear models. The Pythagorean law of information identities is particularly used to identify the best parsimonious log-linear and logit models, and illustrated with a small contingency data table from a study of the risk factors of ischemic stroke. The selection of concise logit (log-linear) models also facilitates the finding of the minimum AIC models. **A comparison study with two existing methods on variable and model selection is thoroughly illustrated along with the construction of logistic regression models using a data of moderate dimension. It appears that all three methods yield similar parsimonious model selection, but differential effects of prediction accuracy may vary with the data under study.**

Keywords: Information identity, logit model, log-linear model, mutual information.

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1. Introduction

In the family of generalized linear models (GLMs), log-linear and logit models have been commonly useful for the inference of categorical variables.¹⁻⁴ It is widely recognized that parsimonious log-linear models and logit models with succinct data interpretation are most desirable in application. There have been review studies comparing subset selection methods in linear regression by assessing the stepwise method coupled with the AIC and BIC criteria along with the likelihood ratio (LR) test against other competing analyzers such as the Bayesian model averaging, penalized regression and elastic net methods for simultaneously selecting variables and estimating regression coefficients.^{5, 6} In general, the stepwise procedure selects variables that are less stable across bootstrap samples compared with other methods, while different methods suggest overlapped subsets of variables and those selected most often between methods tend to be intuitively appealing. Although model fitting and subset selection methods in linear regression can be generalized to categorical variables, there is a paucity of research considering the discrete nature of categorical variables as well as their interactions by making inference complied with discrete probability distributions. The primary purpose of this study is proposing a constructive procedure for modeling categorical variables for log-linear and logit models based on the mutual information (MI) among variables along with the LR statistic in discrete multinomial distributions. The proposed procedure considers simultaneously variable selection and model fitting with particular emphasis on the selection and estimation of interaction effects.

Analogous to the ANOVA decomposition, a log-linear model defines the logarithms of expected cell counts of categorical variables in a contingency table as a linear regression equation of marginal and interaction effects. The logit regression model, as a special case of the log-linear model, predicts the odds of a binary or multinomial response variable of interest (hereafter the target) by considering other variables in the table as predictors. A recent study showed that the MI presents a geometric interpretation of the association between categorical variables, complied with the invariant Pythagorean laws for testing independence against alternative hypotheses in 2-way tables.⁷ An essential extension of the theory to 3-way tables further characterized the geometry for testing conditional independence between two variables given the 3rd one as the hypotenuse of a right triangle whose two legs together define independent tests for the interaction and the

partial association, respectively, among the three variables.⁸ In multivariate tables, the MI of a random vector can be decomposed into lower-dimensional MI terms plus conditional mutual information (CMI) terms which are orthogonal to each other. There are different ways of partitioning the MI and CMI terms for a given table, resulting in different forms of information identities, each characterizing specific associative relations among variables.⁹ However, the total amount of MI shared by the MI and CMI terms are the same across identities.

Parsimonious log-linear or logit models provide simple and intuitive interpretation of data, a condition which has encouraged model-data-fitting in applications by considering only main effects in the model. However, interaction effects are natural building blocks of the log-linear and logit models especially in medical applications.²⁻⁴ For instance, drug-drug interactions and effects of comorbidity on drug efficacies are often considered in data analysis.¹⁰ Moreover, gene-gene and gene-environment interaction effects on diseases have been recognized as central for analyzing gene expression data.¹¹ The interaction between two predictors implies “moderation” in that the effect of one predictor on the target is differentially moderated by the other. Testing and interpretation of interaction effects in logistic regression was fully discussed;¹² these issues are inevitably related to the methodology in model-data-fitting and subset selection. The AIC criterion was developed based on the principle of prediction accuracy compatible to the principle of cross validation, and has been supported together with the AIC_c, BIC in most statistics software packages.^{13, 14} In addition to the stepwise procedure, there is a demand for a systematic and effective assessment tool for constructing a parsimonious log-linear or logit model comprising both main and interaction effects based on the minimum AIC.

In this study, we propose an information theoretical approach to constructing log-linear and logit models through identifying the indispensable interaction effects among variables and their main effects. The study is layed out as follows. In Section 2, a review is devoted to the basic elements of statistical information theory; that is, the MI, CMI, and Pythagorean laws for testing conditional independence in 3-way tables, along with their associations with main and interaction effects in a log-linear model. The information identity defined in a 3-way table is formulated with extensions to multi-way tables. In Section 3, a dataset collected in a clinical study on the risk factors of ischemic stroke is applied to illustrate the theory and methodology described in Section 2. A focus is placed on the information approach to constructing log-linear models by testing interactions

among variables in the table. A sequence of tests on indispensable interaction effects among the variables is carried out such that the most parsimonious log-linear model can be selected. In Section 4, the results from the analysis of log-linear models can be used to facilitate the selection of logit models for the same dataset. The MI analysis is used to first identify indispensable predictors for the target using a similar CMI analysis in the log-linear models. Given the selected predictors, interaction effects of the most parsimonious log-linear model are used to identify the desired interactions in a logit model. This altogether furnishes the entire scheme of constructing both log-linear and logit models based on the geometric MI analysis. As a byproduct, the minimum AIC logit model using the same predictors can be easily identified in a neighborhood of the acquired MI logit model. We conclude the study with a brief discussion of potential extensions of the geometric information analysis to GLMs involving categorical and continuous variables.

2. Information Identity and Log-linear Model

The classical studies of partitioning chi-squares in a 3-way table¹⁵⁻¹⁷ inspired the development of the log-linear model,^{18, 19} which was subsequently used for decades to measure associations among categorical variables.^{2, 20, 21} In this section, we demonstrate the link between basic log-linear models and corresponding information identities in the 3- and multi-way tables.

Let (X, Y, Z) denote a 3-way $I \times J \times K$ contingency table with the joint probability density function (pdf) $f_{X,Y,Z}(i, j, k)$, for $i = 1, \dots, I, j = 1, \dots, J$ and $k = 1, \dots, K$. The Shannon entropy defines a basic equation in terms of joint and marginal probabilities as

$$H(X) + H(Y) + H(Z) = I(X, Y, Z) + H(X, Y, Z). \quad (1)$$

Here,

$$H(X, Y, Z) = -\sum_{i,j,k} f_{X,Y,Z}(i, j, k) \cdot \log[f_{X,Y,Z}(i, j, k)]$$

is the joint entropy,

$$H(X) = -\sum_i f_X(i) \cdot \log[f_X(i)]$$

is the marginal entropy of X, and

$$I(X, Y, Z) = \sum_{i,j,k} f_{X,Y,Z}(i, j, k) \cdot \log \left\{ \frac{f_{X,Y,Z}(i,j,k)}{f_X(i)f_Y(j)f_Z(k)} \right\}$$

denotes the MI between the three variables.^{22, 23} There is a geometric aspect of the MI, which defines the Kullback-Leibler divergence from the joint pdf to the space of products of marginal pdfs, that is, the space of the null hypothesis of independence.^{7, 24} By factoring the joint log-likelihood, an orthogonal partition of the MI among the 3 variables can be expressed as the following information identity

$$I(X, Y, Z) = I(X, Z) + I(Y, Z) + I(X, Y|Z). \quad (2)$$

The right-hand side of (2) admits three equivalent identities by exchanging the common variable Z with either X or Y. Here, a 2-way MI term such as $I(X, Z)$ is defined with the marginal (X, Z) table using an analog of the 3-way table in (1). The conditional mutual information (CMI) $I(X, Y|Z)$ in (2) defines the expectation of the log-likelihood ratio for testing the conditional independence between X and Y across levels of Z. Based on the multivariate multinomial likelihood, Equation (2) and its sample version are valid with the same formula, that is, the same equation holds when the MIs and CMIs are replaced by their sample analogs. In practice, the sample version of (2) can be expressed as

$$\begin{aligned} \hat{I}(X, Y, Z) &= 2N \sum_{i,j,k} \hat{f}_{X,Y,Z}(i, j, k) \log \left(\frac{\hat{f}_{X,Y,Z}(i,j,k)}{\hat{f}_X(i)\hat{f}_Y(j)\hat{f}_Z(k)} \right) \\ &= 2N \sum_{i,k} \hat{f}_{XZ}(i, k) \log \left(\frac{\hat{f}_{XZ}(i,k)}{\hat{f}_X(i)\hat{f}_Z(k)} \right) + 2N \sum_{j,k} \hat{f}_{YZ}(j, k) \log \left(\frac{\hat{f}_{YZ}(j,k)}{\hat{f}_Y(j)\hat{f}_Z(k)} \right) \\ &\quad + 2N \sum_k \left[\sum_{ij} \hat{f}_{XY|Z}(i, j|k) \log \left(\frac{\hat{f}_{XY|Z}(i,j|k)}{\hat{f}_{X|Z}(i|k)\hat{f}_{Y|Z}(j|k)} \right) \right] \\ &= \hat{I}(X, Z) + \hat{I}(Y, Z) + \hat{I}(X, Y|Z), \end{aligned} \quad (3)$$

where N is the total sample size, and the constant 2N is used for the approximations to suitable chi-square distributions. The notation $\hat{f}_{XZ}(i, k)$ denotes the estimated joint pdf in the (i, k) cell, and $\hat{f}_X(i)\hat{f}_Z(k)$ is the estimated product pdf under the assumption of

independence. Other notations in (3) are defined by analogy. It follows that $\hat{I}(X, Z)$, $\hat{I}(Y, Z)$ and $\hat{I}(X, Y|Z)$ on the right-hand side of (3) are asymptotically chi-square distributed with $(I-1)(K-1)$, $(J-1)(K-1)$ and $(I-1)(J-1)K$ degrees of freedom (*df*), respectively.^{9, 18}

In application, the sample MI, denoted by $\hat{I}(X, Y)$, is the LR deviance statistic for testing for 2-way independence between X and Y, which is the same test for the hypothesized log-linear model, denoted by $\{X, Y\}$, composed of the intercept plus the two main effects X and Y. The hypothesis of conditional independence in a 3-way table defines the null CMI, $I(X, Y|Z) = 0$, which also defines the hierarchical log-linear model $\{XZ, YZ\}$. The 3-way Pythagorean law (P-law) depicts that $I(X, Y|Z)$ is the hypotenuse of a right triangle with two orthogonal legs: the interaction effect $\{XYZ\}$, denoted by $Int(X, Y, Z)$, measuring the heterogeneous association between X and Y across the levels of Z, and the partial association, denoted by $Par(X, Y|Z)$, measuring the homogeneous association between X and Y across the levels of Z. Specifically, the CMI term on the right-hand side of (2) is expressed as the sum of two orthogonal components:

$$I(X, Y|Z) = Int(X, Y, Z) + Par(X, Y|Z). \quad (4)$$

Similar to (2) and (3), the sample analogs of these terms in (4) also satisfy the identity in (4). The sample CMI $\hat{I}(X, Y|Z)$ is the MLE of conditional independence, which is the last summand in (3). The MLE $\widehat{Int}(X, Y, Z)$ in (4) can be computed using the iterative proportional fitting or the Newton-Raphson procedure,² and the MLE of the partial association $\widehat{Par}(X, Y|Z)$ can be obtained by the difference between $\hat{I}(X, Y|Z)$ and $\widehat{Int}(X, Y, Z)$. The Pythagorean law in (4) characterizes $I(X, Y|Z)$ to be the hypotenuse of a right triangle with two legs: the interaction $Int(X, Y, Z)$ and the uniform association $Par(X, Y|Z)$. It was proved that the LR statistic $\hat{I}(X, Y|Z)$, testing conditional independence using the chi-square distribution with $(I-1)(J-1)K$ *df*, can be decomposed into a two-step LR test. The first step directly tests the hypothesis of no interaction between X and Y across the levels of Z using $\widehat{Int}(X, Y, Z)$ with $(I-1)(J-1)(K-1)$ *df*, and, only if this hypothesis is accepted, the hypothesis of uniform association is tested using $\widehat{Par}(X, Y|Z)$ with $(I-1)(J-1)$ *df*. In applications, it implies that, given a significant test for the said conditional independence at the usual level 0.05, the hypothesis of no interaction is legitimately tested against a smaller level than 0.05 (cf. Figure 1 in Cheng et al.⁸).

The extension of (2) to multi-way tables leads to information identities in general cases. For instance, the association between a variable T and three predictors X, Y and Z can be measured using the following mutual information (MI) identity:

$$\begin{aligned}
 I(T, X, Y, Z) &= I(X, Y, Z) + I(\{X, Y, Z\}, T) \\
 &= I(X, Z) + I(Y, Z) + I(X, Y|Z) \\
 &\quad + [I(T, Z) + I(T, Y|Z) + I(T, X|\{Y, Z\})]. \quad (5)
 \end{aligned}$$

By using an additional variable T in (3), the term $I(\{X, Y, Z\}, T)$ in (5) is used to measure the association between T and $\{X, Y, Z\}$. If T denotes the target in a logit model, the terms in the brackets of (5) would describe the regression of T on X, Y and Z. For instance, if the null hypothesis CMI $I(T, X|\{Y, Z\}) = 0$ is retained, then X is dispensable while Y and Z are already in the model. With a vector variable Z, equation (5) allows the prediction of T by more than three variables. Equivalent and useful MI identities to (5) can be obtained by interchanging X and Y, or X and Z (cf. Cheng et al.⁹).

3. Mutual Information Log-linear Modeling

Analysis of hierarchical log-linear models for a contingency table usually begins with testing model-data-fit of all two-way interactions among the variables, followed by adding three- and higher-way interactions as needed for fit.^{2, 3, 25-27} In this section, the geometric MI approach is proposed for constructing log-linear models in a multi-way table as an alternative to the conventional analysis. The method is based on testing and deleting dispensable higher-order MI and interaction terms based on the information identities of the full saturated model, and processed step-by-step using the two-step LR test.⁸ It is remarkable that each full-data information identity is MI-equivalent to another identity. Thus, in principle, the method would identify the same indispensable and significant main and interaction effects in the contingency table. It follows that a scheme of constructing parsimonious log-linear models using the least main effects and two- and high-way interactions can be developed. This will be illustrated using the clinical dataset on the risk factors of ischemic stroke.

In the dataset, the computed brain tomography scans were available from 354 patients diagnosed with ischemic stroke in the middle cerebral arterial (MCA) territory

and 1,518 control subjects.²⁸ The data were collected during 2006 through 2008 to investigate the association between ischemic stroke and its risk factors using a logit model, and the calcification burden in the MCA territory was of main interest. The target response was the status of stroke patients versus controls (S: 1 = case; 0 = control), and the risk factors consisted of the calcification burden (C: 1 = yes; 0 = no) in the MCA territory, age (A: 1 \geq 60; 0 < 60), gender (G: 1 = male; 0 = female), hypertension (H: 1 = SBP > 140mm Hg or DBP > 90 mm Hg; 0 = none), diabetes mellitus (D: 1 = fasting serum glucose level > 7.8 mmol/L; 0 = otherwise), and smoking (M: 1 = smoking over 1 cigarette/day; 0 = none). The goal is to assess the best parsimonious log-linear model for the 7-way contingency table.

Assume tentatively all the variables are useful and have at least one significant interaction effect with others. We proceed by inspecting as many dispensable higher-order interactions among the variables as possible, and identify the factors in the order of giving the most insignificant high-way interactions. After deleting insignificant higher-way CMI terms and keeping significant lower-order ones, the scheme continues with inspecting the next insignificant high-way CMI effects among the remaining factors, and stops when all lower-order interactions are significant.

The CMI statistics among the 7 factors indicate that the risk factor C “calcification burden” gives the least significant association with the other 6 factors. By deleting the insignificant CMI terms and keeping the significant ones, the factor C is put aside, and the scheme finds the next risk factor having the most insignificant CMI terms is M, followed by the factor G. Then, the remaining four factors {S, A, D, H} are found highly associated with each other without insignificant CMI terms. In technical terms as equation (5), the basic information identity among the seven risk factors can be expressed as

$$\begin{aligned}
& \hat{I}(C, M, G, S, D, H, A) \\
&= \hat{I}(\{A, D, G, H, M, S\}, C) + \hat{I}(\{A, D, G, H, S\}, M) + \\
& \hat{I}(\{S, A, D, H\}, G) + \hat{I}(\{D, A, H\}, S) + \hat{I}(D, A, H).
\end{aligned} \tag{6}$$

The terms on the right-hand side of equation (6) are orthogonal to each other and their information analysis will be provided in order. The first summand is the MI between C and the other six factors, which is decomposed as the sum of six CMI terms as follows.

$$\begin{aligned}
\hat{I}(\{A, D, G, H, M, S\}, C) &= \hat{I}(C, M|\{A, D, G, H, S\})^* (= 15.232, df = 32, p = .995) \\
&+ \hat{I}(C, G|\{S, H, D, A\})^* (= 9.768, df = 16, p = .878) \\
&+ \hat{I}(C, D|\{S, H, A\})^* (= 5.623, df = 8, p = .689) \\
&+ \hat{I}(C, H|\{S, A\})^* (= 5.057, df = 4, p = .281) \\
&+ \hat{I}(C, A|S) (= 31.449, df = 2, p < 0.001) \\
&+ \hat{I}(C, S) (= 96.972, df = 1, p < 0.001), \tag{7}
\end{aligned}$$

where insignificant ones are marked with asterisks. In short, equation (7) is expressed as

$$\begin{aligned}
\hat{I}(\{A, D, G, H, M, S\}, C) &= \hat{I}(C, \{M, G, D, H\}|\{A, S\})^* (= 35.68, df = 60, p \cong .995) \\
&+ \hat{I}(C, \{A, S\}) (= 128.421, df = 3, p < 0.001). \tag{8}
\end{aligned}$$

From the P-law decomposition of (4), it is seen that $\widehat{Int}(C, A, S) (= 8.234, df = 1, p = .004)$ offers a significant component of $\hat{I}(C, A|S)$.

Next, with the factor M, a similar decomposition of the second summand in (6) is

$$\begin{aligned}
\hat{I}(\{A, D, G, H, S\}, M) &= \hat{I}(M, A|\{D, G, H, S\})^* (= 25.325, df = 16, p = .064) \\
&+ \hat{I}(M, D|\{G, H, S\})^* (= 12.589, df = 8, p = .127) \\
&+ \hat{I}(M, H|\{G, S\})^* (= 5.196, df = 4, p = .268) \\
&+ \hat{I}(M, S|G) (= 16.935, df = 2, p < 0.001) \\
&+ \hat{I}(M, G) (= 314.210, df = 1, p < 0.001). \tag{9}
\end{aligned}$$

Equation (9) can also be expressed as

$$\begin{aligned}
\hat{I}(\{A, D, G, H, S\}, M) &= \hat{I}(M, \{A, D, H\}|\{G, S\})^* (= 43.11, df = 28, p \cong .035) \\
&+ \hat{I}(M, \{G, S\}) (= 331.145, df = 3, p < 0.001), \tag{10}
\end{aligned}$$

where the first summand is insignificant by the Bonferroni or FDR correction level ($\cong .05/4$) against five independent summands in (6). A significant interaction $\widehat{Int}(M, G, S) (= 7.224, df = 1, p = .007)$ is a notable component of the last term in (10).

Similar to equations (8) and (10), the factor G “gender” of the third summand in (6) yields some insignificant and significant CMI and MI terms as

$$\begin{aligned}
\hat{I}(\{S, A, D, H\}, G) &= \hat{I}(G, S|\{A, D, H\})^* (=11.388, df = 8, p = .181) \\
&+ \hat{I}(G, H|\{A, D\})^* (= 8.695, df = 4, p = .069) \\
&+ \hat{I}(G, D|A) (= 18.891, df = 2, p < 0.001) \\
&+ \hat{I}(G, A) (= 13.714, df = 1, p < 0.001) \\
&= \hat{I}(G, \{S, H\}|\{A, D\})^* (= 20.083, df = 12, p = .072) \\
&+ \hat{I}(G, \{A, D\}) (= 32.605, df = 3, p < 0.001). \tag{11}
\end{aligned}$$

The last summand of equation (11), $\widehat{Int}(G, D, A)$ ($= 13.529, df = 1, p < 0.001$), is a significant interaction component.

The association of log-linear models will be complete by analyzing the information of the last two terms in (6), which consist of significant CMI components in the following two equations

$$\begin{aligned}
\hat{I}(\{D, A, H\}, S) &= \hat{I}(S, D|\{A, H\}) (= 22.368, df = 4, p < 0.001) \\
&+ \hat{I}(S, H|A) (= 71.886, df = 2) \\
&+ \hat{I}(S, A) (= 88.586, df = 1), \tag{12}
\end{aligned}$$

and

$$\begin{aligned}
\hat{I}(D, A, H) &= \hat{I}(A, H) (= 228.002, df = 1) \\
&+ \hat{I}(D, H) (= 144.473, df = 1) \\
&+ \hat{I}(D, A|H) (= 36.956, df = 1), \tag{13}
\end{aligned}$$

which comprise significant interaction terms $\widehat{Int}(S, D, \{A, H\})$ ($= 19.690, df = 3$), $\widehat{Int}(S, A, H)$ ($= 13.543, df = 1$) and $\widehat{Int}(A, D, H)$ ($= 16.797, df = 1$). By collecting significant CMI terms in (8), (10), (11), (12) and (13), useful terms in (6) are summarized to yield the approximate mutual information decomposition as

$$\begin{aligned}
\hat{I}(C, M, G, S, D, H, A) &\cong \hat{I}(C, A|S) + \hat{I}(C, S) + \hat{I}(M, S|G) + \hat{I}(M, G) \\
&+ \hat{I}(G, D|A) + \hat{I}(G, A) + \hat{I}(S, A) + \hat{I}(S, H|A) \\
&+ \hat{I}(S, D|\{A, H\}) + \hat{I}(A, D, H). \tag{14}
\end{aligned}$$

Using standard notations of hierarchical log-linear models, a crude summary of equation (14) gives a tentative log-linear model

$$\text{LLM}_1 = \{\text{ACS, GMS, ADG, SADH}\} \quad (15)$$

with residual deviance 92.259 ($df = 99$, $p = .671$), and estimated AIC = 470.981. It is seen that model LLM_1 of (15) may not be parsimonious with a fairly large p value (.671) due to using overlapped 3-way interaction terms such as {SDA, SDH} in the 4-way interaction term SADH. Meanwhile, aside from the common interaction terms, distinct interaction terms expressed by {ACS, GMS, ADG} in (8), (10) and (11) appear to have slightly smaller estimates of mutual information than those of {SDA, SAH, SDH, ADH} in (12) and (13). This refers to the larger estimates such as $\hat{I}(S, H) = 105.425$, $\hat{I}(A, H) = 228.002$, $\hat{I}(D, H) = 144.473$, $\widehat{Int}(S, D, A) = 27.84$, $\widehat{Int}(S, D, H) = 13.571$ and $\widehat{Int}(A, D, H) = 16.797$.

It is well known that hierarchical log-linear models of all 2-way terms (or many 3-way terms) would share overlapped mutual information with 3-, 4- and 5-way terms, respectively, when there are four or more variables. A goal of the current analysis is to emphasize that hierarchical log-linear models be built by using interaction terms without overlapped mutual information. This has been the prerequisite rule of mutual information analysis in the derivation of formulas (6) to (14). Thus, to acquire a more concise model than LLM_1 of (15), it is convenient to select the 2-way terms {CA, CS} from (8), {MS, MG} from (10), and {GA, GD} from (11); and naturally select from (12) three 2-way terms {SA, SH, SD} because $\hat{I}(S, H) (= 105.425, df = 1)$ and $\hat{I}(S, D) (= 24.083, df = 1)$. Next, omit the 3-way interaction term ADG tentatively, and, select two 3-way terms {SDA, SDH} from the first line of (12). Then, select from equation (13) all three significant 2-way terms {AD, AH, DH} and the 3-way term {ADH}, where $\hat{I}(A, D) (= 66.977, df = 1)$, $\hat{I}(A, H) (= 228.002, df = 1)$, and $\hat{I}(D, H) (= 144.473, df = 1)$. Finally, omit the 4-way term SADH of LLM_1 , in which the significant MI measure of $\widehat{Int}(S, D, AH) (= 16.69, df = 3)$ in (12) is only partially reduced due to keeping the 3-way interaction terms {SDA, SDH, ADH}. In summary, we obtain a more concise log-linear model than LLM_1 as

$$\text{LLM}_2 = \{\text{CA, CS, MG, MS, GA, GD, SDA, SDH, ADH}\}, \quad (16)$$

with residual deviance 128.10 ($df = 105$, $p = .062$), and estimated AIC = 494.822. Alternatively, it can be expected from (12) that replacing the 3-way term SDH by SAH in (16) would yield an equally informative model, that is,

$$\text{LLM}_3 = \{\text{CA, CS, MG, MS, GA, GD, SDA, SAH, ADH}\}, \quad (17)$$

with residual deviance 128.124 ($df = 105, p = .062$), and estimated AIC = 494.846.

It is anticipated from the estimated AIC that, being slightly different, log-linear models (16) and (17) appear to be the best choices for this data analysis. Further selection of valid log-linear models may be examined for a comparison study. For this concern, a few 2- and 3-way terms in (16) or (17) may be changed to yield a slightly different model. For example, if $\{\text{GA, GD, SDA, ADH}\}$ in (16) or (17) are replaced with $\{\text{ADG, SDH}\}$, then, it gives a model with deviance 133.727 ($df = 105, p = .031$) that is slightly lack of fit. But, if the set $\{\text{CA, CS}\}$ in (16) or (17) is also replaced by CAS simultaneously, then a valid model is presented as

$$\text{LLM}_4 = \{\text{CAS, GM, MS, ADG, SAH, SDH}\} \quad (18)$$

with deviance 125.493 ($df = 104, p = .074$) and estimated AIC = 494.216. In terms of model parsimony and the AIC estimates, it is clear that model LLM_4 of (18) is less preferred to LLM_2 of (16) or LLM_3 of (17). To summarize, the above MI analysis of hierarchical log-linear models presents a convenient tool for finding the best parsimonious models LLM_2 and LLM_3 for the 7-way contingency table of the ischemic stroke data.

4. Mutual Information Logit Modeling

It is well known that a logit model can be directly obtained from a log-linear model when a particular variable is the response of interest. It is expected that parsimonious logit models can be derived from similar MI analyses to the previous log-linear modeling in Section 3. In the same empirical study, the obvious target is the ischemic stroke status, defined as $S = 1$ for case, and $S = 0$ for control. Models LLM_2 of (16) and LLM_3 of (17) indicate that the association between G (gender) and S is negligible. It is recommended to delete the most insignificant factors through inspecting a basic MI identity between the target S and the six risk factors. The identity is

$$\hat{I}(\{G, M, H, D, C, A\}, S) = \hat{I}(S, G|\{M, H, D, C, A\})^* (= 28.837, df = 32, p = .627)$$

$$\begin{aligned}
&+ \hat{I}(S, M|\{H, D, C, A\}) (= 26.110, df = 16, p = .052) \\
&+ \hat{I}(S, C|\{H, A, D\}) (= 78.153, df = 8, p < 0.001) \\
&+ \hat{I}(S, \{H, A, D\}) (= 182.841, df = 4, p < 0.001). \quad (19)
\end{aligned}$$

The first summand in (19) confirms that there is no significant effect of G on S, given the other risk factors. The last two summands indicates four major risk factors {C, H, A, D} associated with S. It is worth noting that significant association between S and the factors age, diabetes mellitus and hypertension has been examined in the literature, for instance, in Movahed, Sattur & Hashemzadeh²⁹ and Sowers.³⁰ The third term of (19) strongly supports the concern about the calcification burden in the MCI territory. After removing the factor G, the second term of (19) indicates that the factor M (smoking) is barely significant conditional on the remaining four factors, but a clinical question of special interest is whether smoking is related to the ischemic stroke among the risk factors. Indeed, the second term can be expressed as

$$\begin{aligned}
\hat{I}(S, M|\{H, D, C, A\}) &= \widehat{Int}(S, M|\{H, D, C, A\}) (15.495, df = 15, p = .416) \\
&+ \widehat{Par}(S, M|\{H, D, C, A\}) (10.615, df = 1, p = .001). \quad (20)
\end{aligned}$$

Equation (20) shows significant association between M and S, but little interaction effect with the factors {H, D, C, A}. The MI between S and the five risk factors {C, M, H, A, D} can be decomposed by the rule of identifying the least significant higher-order interaction effects, comparable to the MI analysis of log-linear modelling in Section 3. This yields equation (21) and the list of decomposed MI identity in Table 1 below.

$$\begin{aligned}
\hat{I}(S, \{C, M, H, A, D\}) &= \hat{I}(S, D) + \widehat{Par}(S, A|D) + \widehat{Int}(S, D, A) \\
&+ \widehat{Par}(S, H|\{D, A\}) + \widehat{Int}(S, H, \{D, A\}) \\
&+ \widehat{Par}(S, C|\{H, D, A\}) + \widehat{Int}(S, C, \{H, D, A\})^* \\
&+ \widehat{Par}(S, M|\{C, H, D, A\}) + \widehat{Int}(S, M, \{C, H, D, A\})^*. \quad (21)
\end{aligned}$$

Table 1. Partitioned CMI terms in the MI identity (21).

Orthogonal Components	Conditional Mutual Information	Interaction	Partial Association

	<i>LR</i>	<i>df</i>	<i>p</i>	<i>LR</i>	<i>df</i>	<i>p</i>	<i>LR</i>	<i>df</i>	<i>p</i>
$\hat{I}(S, M \{C, H, D, A\})$	26.110	16	< 0.001	15.495	15	0.416	10.615	1	0.001
$\hat{I}(S, C \{H, D, A\})$	78.153	8	< 0.001	11.963	7	0.102	66.190	1	< 0.001
$\hat{I}(S, H \{D, A\})$	55.444	4	< 0.001	12.257	3	0.007	43.187	1	< 0.001
$I(S, A D)$	103.314	2	< 0.001	27.840	1	< 0.001	75.474	1	< 0.001
$\hat{I}(S, D)$	24.083	1	< 0.001						

By deleting two insignificant higher-order interactions, five partial association (main effect) terms $\{M, C, H, A, D\}$ and two interaction terms $\{AD, HDA\}$ remain in the MI identity (21) and Table 1. Comparable to the log-linear model LLM_2 of (16), the $\{SADH\}$ interaction can also be replaced with lower-order interaction terms in the formulation of the logit model for the target S . The estimated MI logit model is

$$\begin{aligned} & \text{logit} [f(S | C, M, H, A, D)] \\ &= -3.584 + 1.653D + 1.659A - 1.003DA + 1.689H - 0.864AH \\ & \quad - 0.763DH + 0.495M + 2.119C, \end{aligned} \quad (22)$$

which gives the residual deviance 26.651 ($df = 23, p = 0.271$). Here, it is remarkable that in using the same predictors $\{C, M, H, A, D\}$ the minimum AIC model can be found among a few neighbors to the MI model (22). By the optimal parsimony of model (22), it is convenient to find the minimum AIC model among a few additional lower-order interaction effects, say, $\{MA, MD, MH\}$ through computing the AIC estimates (cf. *SAS CATMOD* or *SPSS logistic procedure*). The answer is

$$\begin{aligned} & \text{logit} [f(S | C, M, H, A, D)] \\ &= -3.824 + 1.895D + 1.895A - 1.130DA + 1.664H - 0.749DH - 0.841AH \\ & \quad + 1.180M - 0.652MA - 0.663MD + 2.083C. \end{aligned} \quad (23)$$

A comparison of model interpretation between models (22) and (23) is useful. Model (23) has all significant parameter estimates, it yields log-likelihood -49.324, the minimum AIC estimate 120.649 and the residual deviance 18.973 ($df = 21, p = 0.587$). In contrast, the MI model (22) yields a slightly larger AIC estimate 124.327, and smaller log-likelihood

-53.163. The parameter estimate 0.495M in the MI model (22) is replaced by 1.180M in the AIC model (23), and the latter is farther away from the estimate 0.542M in the under-fitted (unreported) logit model of five isolated main effects {A, C, D, H, M}.

On another aspect of model comparison, it is well known that a minimum AIC model is expected to yield the best prediction accuracy in the principle of cross-validation.³¹⁻³⁵ If model prediction accuracy is assessed by testing goodness-of-fit against the residual deviance, then results of a simulated study under various sampling designs of the raw data are reported in the Supplement – S1. It is not surprising that by using two more interaction parameters {MA, MD} with the same set of five predictors, higher average acceptance rates (prediction accuracy) are acquired with the AIC model (23), as compared to the MI model (22) under each sampling design. However, it is essential that higher prediction accuracy is achieved with the AIC modeling criterion at the cost of losing the independent interpretation of the target S by the factor M.

By keeping the barely significant factor M in equation (19), the above analysis has been aimed at inspecting the potential effect of the risk factor M (smoking) on the target S (stroke). Relaxing this purpose, (19) can be rewritten as

$$\begin{aligned} \hat{I}(\{G, M, H, D, C, A\}, S) &= \hat{I}(S, \{G, M\}|\{H, D, C, A\})^* (= 54.947, df = 48, p = .228) \\ &+ \hat{I}(S, C|\{H, A, D\}) (= 78.153, df = 8, p < 0.001) \\ &+ \hat{I}(S, \{H, A, D\}) (= 182.841, df = 4, p < 0.001). \end{aligned} \quad (24)$$

Now, deleting its first summand of equation (24) is equivalent to deleting the first row of Table 1. By using the last two summands of (24), or the last four rows of Table 1, the MI logit model of interpreting S by the minimal predictor set {A, C, D, H} is

$$\begin{aligned} \text{logit}[f(S | A, C, D, H)] &= -3.457 + 1.369D + 1.652A + 1.710H - 0.868AH - 1.061DA \\ &- 0.786DH + 2.111C. \end{aligned} \quad (25)$$

Model (25) has residual deviance 12.054 ($df = 8, p = 0.149$) and AIC estimate 78.389. It concludes that the only predictor that is capable of interpreting the target S is the isolated variable C with a significant parameter estimate 2.111C. It explains that “the odds ratio of having the stroke condition ($S = 1$) against control ($S = 0$) is $\exp(2.111) = 8.256$, when an individual with the MCA calcification burden ($MCA_{ca} = 1$) is compared to one without it

($MCA_{ca} = 0$)". The acquired estimate $2.111C$ in (25) is close to the estimate $2.119C$ in (22), and also to the unjustified estimate $2.125C$ in the under-fitted logit model of only four main effects $\{A, C, D, H\}$, as compared to the estimate $2.083C$ of the AIC model (23). Finally, it is not surprising that the MI model (25) is also the minimum AIC model, and there is no difference in parameter interpretation and prediction accuracy when the same minimal predictor set $\{C, H, A, D\}$ is used.

5. Methods of Model Selection

6. Comparison of Logistic Models

7. Discussion

In this study, we demonstrate the constructive analysis of log-linear and logit modeling using the geometry of mutual information defined with the multivariate multinomial distributions. The proposed analysis is illustrated using a thorough study of the ischemic stroke contingency data table. It is essential that the CMI analysis is able to identify the main-effect predictors and their significant interaction effects such that the acquired log-linear and logit models are undoubtedly most parsimonious. For a reduced finite dimensional contingency table, the conventional approach to log-linear modeling begins with inspecting two-way association effects and successive testing for higher-order interaction effects. In contrast, the proposed geometric analysis develops a backward selection scheme by deleting dispensable higher-order interaction effects through the CMI analysis. As a counterpart to log-linear modeling, the MI analysis naturally constructs the information approach to logit modeling for the same empirical study. The acquired MI logit models are most parsimonious, which usually differ from the minimum AIC models when using the same finite sets of predictors. In the current data analyses, it is found that the AIC model may use a few additional interaction parameters and yield higher prediction accuracy than does the MI model, but at the cost of losing certain independent parameter interpretability for the data.

It is well known that in the modern analysis of contingency tables, standard methods of variable selection and model selection with log-linear models and logit models are often discussed using AIC, BIC and other penalized criteria. These methods have not provided inference methods of direct identification of both indispensable predictors and interaction parameter effects. We have shown that the constructive selection schemes of predictors and models given by the proposed MI analysis fulfill this purpose before a generalized linear model is selected for use. When both categorical and continuous variables are present in the data, our analysis recommends that multivariate multinomial distributions can be employed to characterize the marginal distributions of the continuous variables by discrete approximations. That is, discretized multivariate histograms of the continuous variables can be analyzed to provide an MI analysis of all the variables. This gives the same analysis of indispensable predictors and interaction effects as the log-linear and logit modeling of a contingency table. It is thus expected that the proposed MI analysis can be applied to standard GLMs using both discrete and continuous variables, which will be examined in a future study.

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Supplementary Materials – S1.

As illustrated in Section 4, a simulation study of 10,000 replicates of various sample sizes is conducted to compare the validity of the two models by goodness-of-fit test under three sampling designs. The first design assumes sampling under the MI model (22) or the AIC model (23). The second design assumes sampling under the raw data multinomial distribution with replacement, which is regarded a restrictive design not generally useful. The third design assumes sampling random subsets of the raw data without replacement. Simulation results of testing for model fit against the MI and AIC models under the assumed sampling designs are reported in Table 2 using two sample sizes, 800 and 1000.

Table 2. Proportions of accepting model (22) or (23) under sampling designs

Tested models /sample size	True MI model (22)	True AIC model (23)	Raw data multinomial distribution	Raw data random subsets
MI (22) / 800	.9959	.9780	.8733	.9827
AIC (23) / 800	.9961	.9978	.9487	.9961
MI (22) / 1000	.9954	.9639	.7867	.9804

AIC (23) / 1000	.9959	.9955	.9040	.9971
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Remark: This is the 2nd version of this paper. The 1st version (manuscript) was rejected by *Statistical Methods Medical Research* without review comments in 2017.