TESTING GENERALIZED LORENZ DOMINANCE

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*This version is based on the thesis of Master of Science of Mr. Li-De Ho at the Department of Applied Mathematics, National Donghwa University, June 1998.

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INTRODUCTION

Consider two random variables X and Y, where $X \geq 0$, $X \sim F$; $Y \geq 0$, $Y \sim G$, F and G denote the cumulative distribution functions of X and Y, respectively.

Definition 1 *F Stochastically Dominates G in the Second Order, denoted*

$F \underset{2nd}{\geq} G\,(X \underset{2nd}{\geq} Y)$, iff

$$\int_0^x F(u)du \leq \int_0^x G(u)du, \ all\ x \geq 0;$$

equivalently, G Dominates F in the Generalized Lorenz Order, $F \underset{GL}{\leq} G$,

$$\int_0^p F^{-1}(t)dt = GL_F(p) \geq GL_G(p) = \int_0^p G^{-1}(t)dt, \ all\ 0 \leq p \leq 1.$$

Definition 2 *Let $E_F X = m_F$ and $E_G X = m_G$, if*

$$L_F(p) = \frac{1}{m_F}\int_0^p F^{-1}(t)dt \geq \frac{1}{m_G}\int_0^p G^{-1}(t)dt = L_G(p), \ all\ 0 \leq p \leq 1,$$

then G Dominates F in the Lorenz Order, denoted $F \underset{L}{\leq} G$.

LITERATURE BACKGROUND

Lorenz (1905, JASA)**, Gini** (1914, TRIVS)**, Dalton** (1920, EJ)**, Hadar and Russell** (1969, AJER)**, Hanoch and Levy** (1969, RES)**, Atkinson** (1970, JET), **Rothschild and Stiglitz (**1970, JET), **Gastwirth** (1971, Econometrica), **Shorrocks** (1983, Econometrica); **Kaur, Prakasa Rao and Singh** (1994, Economics Theory); **Ho** (1998).

HYPOTHESES TESTING (McFadden, 1989)

$H_0 : \int_0^x F(u)du \geq \int_0^x G(u)du, \ \text{all } x \geq 0,$

$H_1 : \int_0^x F(u)du < \int_0^x G(u)du, \ \text{some } x > 0.$

$Data : X_1, ..., X_m \sim F; Y_1, ..., Y_n \sim G.$

$m = n, \ S_n^* = \max_x S_n(x) = \max_x \sqrt{n} \int_0^x (G_n(u) - F_m(u))\, du.$

PROPOSITION 1. Under H_0, if $F = G$,

(i) $S_n(x) \to N(0, \rho(x)),$

where $\rho(x) = 2\{\int_0^x (x-u)^2 dF(u) - [\int_0^x (x-u) dF(u)]^2\}$

(ii) $3e^{-t^2/8} > P(S_n^* > t) > \frac{1}{4} e^{-t^2/\pi\sigma^2} + O(1/n),$

where $\sigma^2 = Var_F X \ \ (X \leq 1).$

HYPOTHESES TESTING (Kaur, Prakasa Rao, and Singh, 1994)

$H_0 : \int_0^x F(u)du \leq \int_0^x G(u)du, \ all \ x \geq 0,$
$H_1 : \int_0^x F(u)du > \int_0^x G(u)du, \ some \ x > 0.$

FORMULATION OF TEST STATISTICS

$\int_0^x F_m(u)du = \frac{1}{m} \sum_{i=1}^{m} (x - X_i) I[x \geq X_i] = \frac{1}{m} \sum_{i=1}^{m} U_i(x) = \overline{U}(x)$
$\int_0^x G_n(u)du = \frac{1}{n} \sum_{j=1}^{n} (x - Y_j) I[x \geq Y_j] = \frac{1}{n} \sum_{j=1}^{n} V_j(x) = \overline{V}(x)$
$S^2_{m,F} = \frac{1}{m} \sum_{i=1}^{m} \{U_i(x) - \overline{U}(x)\}^2; \ S^2_{n,G} = \frac{1}{n} \sum_{j=1}^{n} \{V_j(x) - \overline{V}(x)\}^2$
Let $C(x) = \int_0^x \{F(u) - G(u)\}du$, $C_{m,n}(x) = \int_0^x (F_m - F)du - \int_0^x (G_n - G)du$,
$\int_0^x (F_m - G_n) = C_{m,n}(x) + C(x), \ D^2_{m,n}(x) = \frac{1}{m} S^2_{m,F}(x) + \frac{1}{n} S^2_{n,G}(x).$

TEST STATISTICS: $Z_{m,n}(x) = \{C_{m,n}(x) + C(x)\} \diagup D_{m,n}(x)$,

Reject H_0, iff $\inf\limits_{x>0} Z_{m,n}(x) > Z_\alpha$,

where Z_α is the *upper* αth quantile of $N(0,1)$ distribution.

PROPOSITION 2. (KPRS, 1994)

Let $D^2(x) = \frac{1}{\lambda} Var[U_1(x)] + \frac{1}{1-\lambda} Var[V_1(x)]$, $m/(m+n) \to \lambda$

as $m, n \to \infty$, *then*

(i) $\limsup\limits_{m,\, n \to \infty} P\{\inf_x Z_{m,n}(x) > Z_\alpha\} \le \alpha$;

(ii) if $\int_0^{x_0} \{F(u) - G(u)\} du = 0$ *and* $\int_0^x \{F(u) - G(u)\} du > 0$, *all* $x \neq x_0$,

then $\lim P\{\inf_x Z_{m,n}(x) > Z_\alpha\} = \alpha$,

(iii) for $(F, G) \in H_1$, $P\{\inf_x Z_{m,n}(x) > Z_\alpha\} \to 1$.

THE PROPOSED TEST

$H_0 : \int_0^x F(u)du \geq \int_0^x G(u)du,\ all\ x \geq 0,$

$H_1 : \int_0^x F(u)du < \int_0^x G(u)du,\ some\ x > 0.$

Under H_1, $C(x_0) = \inf_x C(x) = \inf_x \int_0^x (F - G)$ exists.

Let $\widehat{C}(x) = \int_0^x (F_m - G_n)du = C_{m,n}(x) + C(x) \simeq \int_0^x (F - G) = C(x)$, as m, n large,

since $\| C_{m,n}(x) \|_\infty = O((\log\log m)/m)^{1/2}$ w.p. 1.

PROPOSITION 3. $\sqrt{m+n}\, C_{m,n}(x) \xrightarrow{d} W(x),\ x > 0,$

$$W(x) = \int_0^x \{\tfrac{1}{\sqrt{\lambda}} B_1(F(u)) - \tfrac{1}{\sqrt{1-\lambda}} B_2(G(u))\} du,$$

where B_1, B_2 *are independent Brownian Bridge Processes, and*

$$Var[W(x)] = \tfrac{1}{\lambda} \int_0^x \int_0^x [F(u) \wedge F(v) - F(u) F(v)]\, dudv + \tfrac{1}{1-\lambda} \int_0^x \int_0^x [G(u) \wedge G(v) - G(u) G(v)]\, dudv.$$

Let $(m+n)\,D^2_{m,n}\,(x) = Var[C_{m,n}(x)] \rightarrow$

$$\tfrac{1}{\lambda}Var[U_1(x)] + \tfrac{1}{1-\lambda}Var[V_1(x)] \equiv D^2(x).$$

Let $\widehat{C}(x_0) = \inf_x \widehat{C}(x) \simeq C(x_0)$.

Then, $\frac{\widehat{C}(x_0)}{D_{m,n}(x_0)} = \frac{C_{m,n}(x_0)}{D_{m,n}(x_0)} + \frac{C(x_0)}{D_{m,n}(x_0)} \overset{d}{\simeq} N(0,1) + \frac{\sqrt{m+n}C(x_0)}{D(x_0)}$

THEOREM 1. *(Ho and Cheng, 1998) Under H_0, $F = G$, $C\,(x_0) = 0$, the approximation in distribution $\frac{\widehat{C}(x_0)}{D_{m,n}(x_0)} \overset{d}{\simeq} N(0,1)$,*
holds such that $P\left(\frac{\widehat{C}(x_0)}{D_{m,n}(x_0)} > Z_\alpha\right) \rightarrow \alpha$, equivalently,

$P\left\{\| \frac{\widehat{C}(x_0)}{D_{m,n}(x_0)} \| \left(\overset{d}{\simeq} |N(0,1)|\right) > Z_{\alpha/2}\right\} \rightarrow \alpha,$ *as m, n large, where*

Z_α is the upper αth quantile of the standard normal distribution.

Under H_1, $C(x_0) = \inf_x C(x) < 0$, $\sqrt{m+n}\;(C(x_0)/D(x_0)) \rightarrow -\infty$,

so that $P\left\{\frac{\widehat{C}(x_0)}{D_{m,n}(x_0)} \left(\overset{d}{\simeq} N(0,1)\right) + \frac{\sqrt{m+n}C(x_0)}{D(x_0)} < Z_\alpha\right\} \rightarrow 1.$

RATIONALE AND FUTURE STUDY

PROPOSITION 4. *Fellman (1976, Econometrica)*

If $Y = g(X)$, $g \nearrow$, $g(x)/x \nearrow$ *or* $\searrow$;

equivalently, $G^{-1}(t)/F^{-1}(t) \nearrow$ *or* $\searrow$;

then, $F \underset{L}{\leq} G$ *or* $F \underset{L}{\geq} G$.

PROPOSITION 5. *Lambert (1989, Chapter 3)*

If GL_F *crosses* GL_G *a finite number of times, the first from above, then*

(i) F *crosses* G*, first from below;*
(ii) if $m_F < m_G$, GL_F *crosses* GL_G *an odd number of times;*
(iii) if $m_F > m_G$, GL_F *crosses* GL_G *an even number of times;*
(iv) if $m_F = m_G$, GL_F *crosses* GL_G *an even/odd number of times, iff* F *crosses* G *an odd/even number of times.*

REMARK. It is understood that general crossing conditions of the generalized Lorenz curves are complicated to be tested in theory. It is also not an easy task to derive an effective test for Lorenz dominance in computing the asymptotic $p - values$, even if it can be achieved theoretically as an extension of Theorem 1.

*This study is based on the thesis of Mr. Li-De Ho, as a partial fulfilment for the degree of Master of Science, Department of Applied Mathematics, National Donghwa University, Hualien, Taiwan, June 1998.

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